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(2+1) Dimensional Black Hole and (1+1) Dimensional Quantum Gravity

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Abstract

In the Chern-Simons gauge theory formulation of the spinning (2+1) dimensional black hole, we may treat the horizon and the spatial infinity as boundaries. We obtain the actions induced on both boundaries, applying the Faddeev and Shatashvili procedure. The action induced on the boundary of the horizon is precisely the gauged $SL(2,R)/U(1)$ Wess-Zumino-Witten (WZW) model, which has been studied previously in connection with a Lorentz signature black hole in (1+1) dimensions. The action induced on the boundary of spatial infinity is also found to be a gauged $SL(2,R)$ WZW model, which is equivalent to the Liouville model, the covariant action for the (1+1) dimensional quantum gravity. Thus, the (2+1) dimensional black hole is intimately related to the quantum gravity in (1+1) dimensions.

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The low dimensional gravity has received much attention as a laboratory to study the interrelation between the space-time geometry and the quantum mechanics since the seminal works on (1+1) dimensional gravity by Teitelboim [1], by Jackiw [2], and by Polyakov [3] and on (2+1) dimensional gravity by Deser, Jackiw, and 't Hooft [4], by Achúcarro and Townsend [5], and by Witten [6]. As the black hole solutions are found in (2+1) dimensional gravity by Bañados, Henneaux, Teitelboim, and Zanelli [7] and in the (1+1) dimensional dilaton gravity by Callan, Giddings, Harvey and Strominger [8], it was understood that the difficult problems associated with the black hole [9] can be dealt in low dimensions in forms far simpler than in (3+1) dimensions. Since then the low dimensional black holes have served as an important arena where one can study the quantum gravity, perhaps the greatest crux in theoretical physics, in a rather tractable manner.

In the Chern-Simons formulation of the (2+1) dimensional black hole, we may treat the horizon as well as the surface of spatial infinity as a boundary. In the present paper, we show that the induced actions on both boundaries for the spinning (2+1) dimensional black hole [7] are given by gauged $SL(2,R)$ Wess-Zumino-Witten (WZW) models [10,11]. However, these two gauged $SL(2,R)$ WZW models are different from each other. (We may get various inequivalent gauged $SL(2,R)$ WZW models, depending on which subgroup is gauged, since the group manifold of $SL(2,R)$ has an indefinite metric.) The induced action on the horizon is found to be equivalent to the action of the Lorentz signature black hole in (1+1) dimensions, which has been discussed extensively by Witten [12]. On the other hand the gauged $SL(2,R)$ WZW action induced on the surface of spatial infinity corresponds to the Liouville action which has been studied previously as the covariant action for the (1+1) dimensional gravity [3,13,14]. It confirms the recent result obtained by Coussaert, Henneaux and Driel [15]. Both cases show that the (2+1) dimensional black hole is intimately related to the quantum gravity in (1+1) dimensions.

The present paper extends the work of Carlip and Teitelboim on the (2+1) dimensional black hole [16,17] and clarifies some issues: The induced boundary actions are *gauged* $SL(2,R)$ WZW models so that the difficulties associated with the nonunitarity of the repre-

sentations of (ungauged) $SL(2,R)$ WZW model can be avoided. Since the (2+1) dimensional black hole has been known to be an exact solution of the effective action of (2+1) dimensional string [18] and dual to a black string, the present paper may shed some light upon these related topics also.

The BTZ (Bañados-Teitelboim-Zanelli) black hole exists

$$
\begin{aligned}
ds^2_{BTZ} &= -N^2dt^2 + N^{-2}dr^2 + r^2(N^{\phi}dt + d\phi)^2, \\
N^2(r) &= -M + \frac{r^2}{l^2} + \frac{J^2}{4r^2}, \\
N^{\phi}(r) &= -\frac{J}{2r^2}
\end{aligned}
\tag{1}
$$

in the presence of a (negative) cosmological constant $\lambda = -1/l^2$. M and J correspond to the mass and angular momentum of the black hole respectively. In (2+1) dimensions the gravity is governed by a Chern-Simons action with an appropriate Lie-algebra valued gauge fields [5,6]. In the presence of a cosmological constant the space-time is asymptotically anti-de Sitter, of which symmetry group is $SO(2,2)$ and the gravity is described by the Chern-Simons action with the $SL(2,R) \otimes SL(2,R) \simeq SO(2,2)$ Lie algebra valued gauged fields

$$
I_{CS}(A,\bar{A}) = \frac{k}{4\pi}\int_M \mathrm{tr}\,\left(AdA + \frac{2}{3}AAA\right) - \frac{k}{4\pi}\int_M \mathrm{tr}\,\left(\bar{A}d\bar{A} + \frac{2}{3}\bar{A}\bar{A}\bar{A}\right) \tag{2}
$$

where $k = -\frac{l}{4G}$, $\quad A = A^aJ_a, \quad \bar{A} = \bar{A}^a\bar{J}_a$ and

$$
[J_a, J_b] = \epsilon_{ab}{}^cJ_c, \qquad \left[\bar{J}_a, \bar{J}_b\right] = \epsilon_{ab}{}^c\bar{J}_c, \qquad \left[J_a, \bar{J}_b\right] = 0.
$$

Here G is the gravitational constant. The equation of motion for the gauge field implies that the gauge field is a pure gauge and may be written in terms of multivalued gauge functions u and $\bar{u}$ as $A = u^{-1}du$, $\bar{A} = \bar{u}^{-1}d\bar{u}$ [19].

The BTZ black hole solution Eq.(1) has two horizons: outer one at $r = r_+$ and inner one at $r = r_-$ where $r_\pm$ are two zeros of $N(r)$. Thus, one may divide the space into three regions bounded by the horizons: $0 < r < r_-$ (Σ_I), $r_- < r < r_+$ (Σ_{II}), $r_+ < r < \infty$ (Σ_{III}). Since the curvature is constant everywhere, it seems ad hoc to divide the space such a way.

However, if we are concerned with the quantum theory using the path integral, it seems unavoidable to confine ourselves to the space-time $M = \Sigma_{III} \times R$. M has two boundaries, ∂M_1 at $r = r_+$ and ∂M_2 at the spatial infinity.

As is discussed in ref. [20], we need to supplement the Chern-Simons gravity action Eq.(2) by boundary terms so that the boundary conditions are imposed consistently. For each boundary, an appropriate boundary term will be introduced. Since we can discuss both boundary actions in a similar way, we will proceed with the boundary action on ∂M_1 first. It is convenient to introduce a new coordinates (τ, ρ, ϕ) to discuss the BTZ black hole in the region M, where $\tau = t/l$ and $r^2 = r_+^2 \cosh^2 \rho - r_-^2 \sinh^2 \rho$. In terms of new coordinates, we see that the BTZ black hole solution has a chiral structure:

$$\begin{cases} A_R = \frac{2}{l}(r_+ - r_-)\left(\sinh\rho J_0 + \cosh\rho J_2\right), \\ A_L = 0, \\ A_\rho = J_1, \end{cases} \tag{3a}$$

$$\begin{cases} \bar{A}_R = 0 \\ \bar{A}_L = -\frac{2}{l}(r_+ + r_-)\left(\sinh\rho \bar{J}_0 - \cosh\rho \bar{J}_2\right), \\ \bar{A}_\rho = -\bar{J}_1. \end{cases} \tag{3b}$$

where $A_{R/L} = A_\tau \pm A_\phi$, and $\bar{A}_{R/L} = \bar{A}_\tau \pm \bar{A}_\phi$. The boundary values of the Chern-Simons gauge fields on ∂M_1 read from the above solution are

$$\begin{aligned} &A_R = \frac{2}{l}(r_+ - r_-)J_2, \quad A_L = 0, \quad A_\rho = J_1, \\ &\bar{A}_R = 0, \quad \bar{A}_L = \frac{2}{l}(r_+ + r_-)\bar{J}_2, \quad \bar{A}_\rho = -\bar{J}_1. \end{aligned} \tag{4}$$

These boundary values suggest us to take the boundary conditions as

$$A_L = 0, \quad \bar{A}_R = 0. \tag{5}$$

(With other choices, the induced actions on the boundaries may suffer unitarity problem.) In order to impose the boundary condition consistently, we should introduce the boundary term as follows

$$I_B = -\frac{k}{4\pi}\int_{\partial M} \mathrm{tr}(A_\tau - A_\phi)A_\phi + \frac{k}{4\pi}\int_{\partial M} \mathrm{tr}(\bar{A}_\tau + \bar{A}_\phi)\bar{A}_\phi, \tag{6}$$

where $\partial M = \partial M_1$. This boundary term is chosen such that its variation cancels that of the Chern-Simons action.

The gauge invariance of the action is now broken, partly because the space-time M has boundaries, and partly because the boundary terms do not respect the gauge invariance. As a consequence, the degrees of freedom of the gauge fields corresponding to the broken symmetry cannot be gauged away any longer. They become dynamical degrees of freedom as Carlip discussed [16]. To get the proper action for these degrees of freedom, we resort to the Faddeev and Shatashvili (FS) proposal for the consistent quantization of anomalous theory [21].

The FS proposal is to introduce a one-cocycle in such a way the local gauge symmetry is restored and to use it as the action describing the "would be" gauge degrees of freedom. This procedure has been applied to construct the action for the (1+1) dimensional quantum gravity [14]. The one-cocycle for the Chern-Simons gravity is constructed to be

$$\begin{aligned}
\alpha_G[A, \bar{A}, g, \bar{g}] &= I_{CS}(A^g, \bar{A}^{\bar{g}}) + I_B(A^g, \bar{A}^{\bar{g}}) - I_{CS}(A, \bar{A}) - I_B(A, \bar{A}) \\
A^g &= g^{-1}dg + g^{-1}Ag \\
\bar{A}^{\bar{g}} &= \bar{g}^{-1}d\bar{g} + \bar{g}^{-1}\bar{A}\bar{g}.
\end{aligned} \tag{7}$$

One sees that $\alpha_G[A, \bar{A}, g, \bar{g}]$ satisfies the one-cocycle condition by construction as usual

$$\delta\alpha_G = \alpha_G[A^h, \bar{A}^{\bar{h}}, g, \bar{g}] - \alpha_G[A, \bar{A}, hg, \bar{h}\bar{g}] + \alpha_G[A, \bar{A}, h, \bar{h}] = 0 \tag{8}$$

and thanks to it, the gauge symmetry $SL(2,R) \otimes SL(2,R)$ is fully restored. The explicit expression for the one-cocycle is

$$\begin{aligned}
\alpha_G(A, \bar{A}, g, \bar{g}) &= \alpha_1(A, g) + \bar{\alpha}_1(\bar{A}, \bar{g}), \\
\alpha_1(A, g) &= \Gamma^L[g] + \frac{k}{2\pi}\int_{\partial M} \mathrm{tr}(\partial_\phi g g^{-1})A_L, \\
\bar{\alpha}_1(\bar{A}, \bar{g}) &= -\Gamma^R[\bar{g}] - \frac{k}{2\pi}\int_{\partial M} \mathrm{tr}(\partial_\phi \bar{g}\bar{g}^{-1})\bar{A}_R,
\end{aligned} \tag{9}$$

$$
\Gamma^L[g] = \frac{k}{4\pi}\int_{\partial M} \mathrm{tr}(g^{-1}\partial_- g)(g^{-1}\partial_\phi g) - \frac{k}{12\pi}\int_M \mathrm{tr}(g^{-1}dg)^3,
$$
$$
\Gamma^R[\bar{g}] = \frac{k}{4\pi}\int_{\partial M} \mathrm{tr}(\bar{g}^{-1}\partial_+ \bar{g})(\bar{g}^{-1}\partial_\phi \bar{g}) - \frac{k}{12\pi}\int_M \mathrm{tr}(\bar{g}^{-1}d\bar{g})^3
$$

where $\partial_\pm = \partial_\tau \pm \partial_\phi$. If ∂M is the boundary of the spatial infinity, one should replace k by $-k$ in the action except for the coefficients of the WZ terms. Eq.(9) shows that the induced action on ∂M_1 is given by a direct sum of two chiral WZW actions: one with left moving chiral boson field only and the other with right moving chiral boson field only. The second terms in $\alpha_1(A, g)$ and $\bar{\alpha}_1(\bar{A}, \bar{g})$ describe coupling of the chiral bosons to the gauge fields. (These terms will be important when we evaluate the black hole entropy.) Due to the Gauss' constraint the gauge fields do not have local degrees of freedom. Once a gauge condition and boundary conditions are chosen appropriately, the gauge fields would be given uniquely by the classical BTZ black hole solution.

Applying the FS proposal to the BTZ black hole system, we find that the quantum action induced on the boundary is described by two chiral WZW models with opposite chirality, which are coupled to the classical BTZ black hole background. Making use of the Polyakov-Wiegman identity, these two chiral fields can be interpreted as left and right moving modes of a non-chiral WZW model [15]

$$
I_1 = \Gamma^L[g] + \Gamma^R[\bar{g}^{-1}] = \Gamma[\bar{g}^{-1}g] \equiv \Gamma[h]. \tag{10}
$$

Therefore, we may conclude that the quantum induced action is given as the non-chiral $SL(2,R)$ WZW model. However, we must note that the gauge symmetry, $SL(2,R) \otimes SL(2,R) \simeq SO(2,2)$ is not completely broken by the classical BTZ black hole solution.

Recall that the three dimensional geometry is completely determined by holonomies or Wilson loops of the Chern-Simons gauge fields [22]

$$
W[C] = \mathcal{P}\exp\left(\oint_C A_\mu dx^\mu\right), \tag{11}
$$
$$
\bar{W}[C] = \mathcal{P}\exp\left(\oint_C \bar{A}_\mu dx^\mu\right)
$$

and the holonomies depend only on the homotopy class of C, where C is a closed curve and $\mathcal{P}$ denotes a path ordered product. We observe that the holonomies do not depend on

the starting point of the curve C on ∂M_1. In the BTZ black hole the only homotopically nontrivial closed curve is C: $\phi(s) = 2\pi s$, $s \in [0,1]$ and all other homotopically nontrivial ones can be given as products of C's. The holonomies transform under gauge transformation as

$$W[C] \to gW[C]g^{-1}, \quad \bar{W}[C] \to \bar{g}W[C]\bar{g}^{-1}. \tag{12}$$

If we take C a space-like closed curve on ∂M_1,

$$\begin{aligned} W[C] &= \exp\left[\frac{2\pi}{l}(r_+ - r_-)J_2\right], \\ \bar{W}[C] &= \exp\left[-\frac{2\pi}{l}(r_+ + r_-)\bar{J}_2\right]. \end{aligned} \tag{13}$$

Considering the following gauge transformation generated by

$$\Lambda = \exp(fJ_2), \quad \bar{\Lambda} = \exp(\bar{f}\bar{J}_2) \tag{14}$$

we find that the holonomies are invariant under the gauge transformation generated by Λ and $\bar{\Lambda}$. It implies that we should equally take $A^{\Lambda}_{cl} = \Lambda^{-1}d\Lambda + \Lambda^{-1}A_{cl}\Lambda$ ($\bar{A}^{\bar{\Lambda}}_{cl} = \bar{\Lambda}^{-1}d\bar{\Lambda} + \bar{\Lambda}^{-1}\bar{A}_{cl}\bar{\Lambda}$) as the classical background on the boundary as well as A_{cl} ($\bar{A}_{cl}$) given by Eq.(3). Taking this into account we may write the path integral representing the generating functional as

$$\begin{aligned} Z &= \int D[\Lambda,\bar{\Lambda}]D[g,\bar{g}] \exp\left\{iI_G(A^{\Lambda},\bar{A}^{\bar{\Lambda}},g,\bar{g})\right\} \\ &= \int D[\Lambda,\bar{\Lambda}]D[g,\bar{g}] \exp\left\{iI_G(A,\bar{A},\Lambda^{-1}g,\bar{\Lambda}^{-1}\bar{g})\right\}, \\ I_G &= I_{CS}(A,\bar{A}) + I_B(A,\bar{A}) + \alpha_G(A,\bar{A},g,\bar{g}). \end{aligned} \tag{15}$$

Here we have chosen appropriate gauge fixing conditions for the gauge fields. Hence, the resultant action is a $U(1)$ gauged WZW model on ∂M and this $U(1)$ gauge group should not be taken into account in construction of $\alpha_G[A,\bar{A},g,\bar{g}]$, which is introduced to restore the gauge invariance. That is, the $U(1)$ subgroup generated by J_2 should be gauged. Then the correct induced quantum action must be a $SL(2,R)/U(1)$ WZW model

$$I_1 = \Gamma[h] + \frac{k}{4\pi}\int_{\partial M} \mathrm{tr}(B_+\partial_- h h^{-1} - B_- h^{-1}\partial_+ h + B_+ h B_- h^{-1} - B_+B_-) \tag{16}$$

where $B_\pm = B_\pm^2 J_2$. This gauged WZW action has been found to depict a Lorentzian black hole in (1+1) dimensions by Witten [12]: The target manifold of the gauged WZW model can be understood as a (1+1) dimensional black hole with a Lorentz signature [23].

Construction of the induced action on the boundary at spatial infinity can be proceed in parallel to that of the induced action on the horizon. As $r \to \infty$ (equivalently, $\rho \to \infty$),

$$\begin{aligned} &A_R \to \frac{e^\rho}{l}(r_+ - r_-)J_+, \quad A_L = 0, \quad A_\rho = J_1, \\ &\bar{A}_R = 0, \quad \bar{A}_L = -\frac{e^\rho}{l}(r_+ + r_-)\bar{J}_-, \quad \bar{A}_\rho = -\bar{J}_1 \end{aligned} \tag{17}$$

where $J_\pm = J_0 \pm J_2$. In order to impose the boundary conditions, same as Eq.(5) on ∂M_2, we should also introduce the boundary terms on ∂M_2 (Eq.(6)). Following the same procedure we applied to the induced action on ∂M_1, we find that the induced action on ∂M_2 can be given also by the $SL(2,R)$ WZW model. However, we should take note of difference in the boundary values. The boundary values of the gauge fields, Eq.(17) have a structure different from that of the boundary values on ∂M_1 so that the subgroup to be gauged is not the same. As a result, we will get a different gauged WZW model as the induced quatum action on ∂M_2.

For a closed curve C on ∂M_2 the Wilson loop elements are

$$\begin{aligned} W[C] &= \begin{pmatrix} 1 & 0 \\ \frac{e^\rho \pi}{l}(r_+ - r_-) & 1 \end{pmatrix}, \\ \bar{W}[C] &= \begin{pmatrix} 1 & \frac{e^\rho \pi}{l}(r_+ + r_-) \\ 0 & 1 \end{pmatrix}. \end{aligned} \tag{18}$$

It follows that gauge group elements that leave the Wilson loop element invariant have the following forms:

$$g = \begin{pmatrix} 1 & 0 \\ f & 1 \end{pmatrix}, \quad \bar{g} = \begin{pmatrix} 1 & \bar{f} \\ 0 & 1 \end{pmatrix}. \tag{19}$$

These subgroups are unbroken on the boundary of spatial infinity and should be factored out in construction of the induced quantum action as before. But in order to gauge these

subgroups we need to take some care. When these subgroups are gauged, the WZW model takes the form different from that of the WZW model for the coset conformal field theory [10], which is the case for the quantum action induced on ∂M_1. Gauging these subgroups can be taken care of by introducing constraints as follows

$$I_2 = \Gamma[h] + \int_{\partial M_2} \mathrm{tr}\Big\{\lambda_1 \left(h^{-1}\partial_+ h J_+ - a\right) + \lambda_2 \left(\partial_- h h^{-1} J_- - b\right)\Big\} \tag{20}$$

where a and b are some appropriate constants. Here $\Gamma[h]$ is defined on ∂M_2. These constraints, which can be rewritten in terms of g and $\bar{g}$

$$\mathrm{tr}\Big\{\lambda_1 \left(g^{-1}\partial_+ g J_+ - a\right) + \lambda_2 \left(\partial_- \bar{g}^{-1} \bar{g} \bar{J}_- - b\right)\Big\} \tag{21}$$

generate the gauge transformation which leaves the Wilson loop elements invariant. As is well known in the study of covariant action for two dimensional gravity, this constrained $SL(2,R)$ WZW model is equivalent to the Liouville model [11,14]. Making use of the Gauss decomposition $h = ABC\omega$,

$$\begin{aligned} A &= \begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} e^{\varphi/2} & 0 \\ 0 & e^{-\varphi/2} \end{pmatrix}, \\ C &= \begin{pmatrix} 1 & 0 \\ w & 1 \end{pmatrix}, \quad \omega = \pm \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \end{aligned} \tag{22}$$

and integrating out the Lagrangian multipliers λ_1 and λ_2, we get the Liouville model

$$\begin{aligned} I_2 &= \frac{1}{8\pi} \int_{\partial M_2} \left(\partial_+ \varphi \partial_- \varphi + (kab) \exp(\beta\varphi)\right), \\ \beta &= \sqrt{\frac{2}{|k|}}. \end{aligned} \tag{23}$$

Thus, we confirm the result of Coussaert, Henneaux and Driel obtained in their study on the (2+1) dimensional black hole [15].

The present paper will be concluded with a few remarks. We obtained the induced actions for the spinning (2+1) dimensional black hole on the boundary which consists of the horizon and the surface of spatial infinity, adopting the Faddeev-Shatashvili procedure,

which yields gauge invariant actions. Resultant induced quantum actions are indentified as $SL(2,R)$ WZW models for both cases. However, we point out that some subgroups of $SL(2,R)$ are unbroken so that the corresponding degrees of freedom should not be taken into account as physical ones. Thus, the correct induced quantum actions are gauged WZW models. For the induced action on the horizon we obtain the $SL(2,R)/U(1)$ WZW model and for that on the spatial infinity, the Liouville model. It is interesting to note that the two dimensional quantum gravity is essential to understand the (2+1) dimensional black hole: the $SL(2,R)/U(1)$ WZW model describes a Lorentz black hole in (1+1) dimensions and the Liouville model can serve as a covariant action for the (1+1) dimensional quantum gravity. Since both induced actions are known exactly soluble, the present paper supports the work of Witten [6], which asserts that the (2+1) dimensional gravity is exactly soluble at the classical and quantum levels. In contrast to the (ungauged) $SL(2,R)$ WZW model, the gauged $SL(2,R)$ WZW models we discussed as the induced quantum actions have unitary representations. So the entropy of the BTZ black hole may be evaluated in a precise manner and the works of Carlip and Teitelboim [16,17] can be improved. We also discussed the asymptotic dynamics of the BTZ black hole and confirmed the result of ref. [15] in the same framework. The present paper shows clearly that the (2+1) dimensional black hole is intimately related to the quantum gravity in (1+1) dimensions.

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[23] In Witten's work the $U(1)$ subgroup to be gauged is generated by J_1. But J_1 can be transformed into J_2 by a global $SL(2,R)$ transformation. Thus, the above gauged WZW model is quivalent to that of [12].